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SOME VARIATIONAL FORMULATIONS FOR BUCKLING ANALYSIS OF CIRCULAR CYLINDERS†

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Abstract—Two new variational functionals are introduced for buckling analysis of cylindrical shells. In the first, admissible functions satisfy all the equilibrium equations and the Euler–Lagrange equations emerge as a set of compatibility equations. In this sense this procedure complements the well known minimum potential energy method. In the second, the transverse equilibrium and membrane compatibility requirements are satisfied by the admissible functions and the system equations emerge as a set of membrane equilibrium and transverse compatibility equations. This formulation can be considered as complementary to Von Karman's well known method. The new functionals are used to obtain solutions for 2 classic examples.

1. INTRODUCTION

In the theory of elasticity variational formulations have provided not only a unifying point of view but they also have facilitated the development of many approximate methods of analysis. In the small deformation theory, a number of interrelated variational functionals have been developed and an excellent discussion of these can be found in the text "Variational Methods in Elasticity and Plasticity—Washizu, 1968". For problems of large deformation analysis invariably the principle of minimum potential energy has been employed for development of approximate methods of analysis and the development of other variational functionals has not received extensive attention.

The large deflection analysis of thin circular cylinders is of considerable technical importance in view of the extensive use of such cylinders in engineering structures. In Ref. [1], four variational formulations for large deformation analysis of thin cylindrical shells were examined for purposes of developing approximate methods of analysis. Of these the first two are well known namely "minimum potential energy" and "Von-Karman" formulations, the remaining two are complementary to these formulations.

In this paper, the two new functionals derived in [1], namely the complementary energy and the complementary Von Karman functionals, are linearized and thereby the corresponding functionals for the linear buckling analysis of circular cylinders are obtained. The application of these functionals is illustrated through some simple examples.

2. BASIC EQUATIONS

For the circular cylindrical shell theory which is of concern to us here, the essential equations may be written as follows:

Strain, curvatures-displacements relations

$$\{\epsilon\} = [D]\{u\} + \tfrac{1}{2}[W]_{,x}\{W\}_{,x} + W\{R\}$$
$$\{\kappa\} = \{W\}_{,xx}. \tag{2.1}$$

Equilibrium equations

$$[D]^T\{N\} = 0$$
$$\{L\}^T\{M\} + (\{W\}^T_{,xx} - \{R\}^T)\{N\} = -\bar{p}. \tag{2.2}$$

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Constitutive equations

$$\{N\} = C[C]\{\epsilon\}, \{M\} = -D[C]\{\kappa\} \tag{2.3}$$

where

$$\{\epsilon\}^T = [\epsilon_{xx} \quad \epsilon_{\theta\theta} \quad \epsilon_{x\theta}], \qquad \{\kappa\}^T = [\kappa_{xx} \quad \kappa_{\theta\theta} \quad \kappa_{x\theta}]$$

$$\{u\}^T = [u_x \quad u_\theta], \qquad \{W\}^T_{,x} = \left[W_{,x} \quad \frac{1}{R}W_{,\theta}\right]$$

$$\{R\}^T = \left[0 \quad \frac{1}{R} \quad 0\right], \qquad \{W\}^T_{,xx} = \left[W_{,xx} \quad \frac{1}{R^2}W_{,\theta\theta} \quad \frac{1}{R}W_{,x\theta}\right] \tag{2.4}$$

$$[D] = \begin{bmatrix} \partial/\partial_x & 0 \\ 0 & \dfrac{1}{R}\dfrac{\partial}{\partial\theta} \\ \dfrac{1}{2R}\dfrac{\partial}{\partial\theta} & \dfrac{1}{2}\dfrac{\partial}{\partial x} \end{bmatrix}, \qquad \begin{bmatrix} W_{,x} & 0 \\ 0 & \dfrac{1}{R}W_{,\theta} \\ \dfrac{1}{R}W_{,\theta} & W_{,x} \end{bmatrix} = [W]_{,x}$$

$$\{L\}^T = \left[\frac{\partial^2}{\partial x^2} \quad \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} \quad \frac{2}{R}\frac{\partial^2}{\partial x\partial\theta}\right],$$

$$\{M\}^T = [M_{xx} \quad M_{\theta\theta} \quad M_{x\theta}], \{N\}^T = [N_{xx} \quad N_{\theta\theta} \quad N_{x\theta}]$$

$$[C] = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 2(1-\nu) \end{bmatrix}, C \equiv \frac{E}{(1-\nu^2)}, D \equiv \frac{Eh^3}{12(1-\nu^2)}$$

and various symbols have their conventional meanings.

A lucid account of the derivation of above equations and their inherent simplifying assumption can be found in Ref. [5, 6]. If along parts of circular cylindrical shell's boundary, denoted by s_u, kinematic conditions are prescribed and along the remaining part s_σ, force quantities are specified, then the boundary conditions of the shell may be stated as follows

Along s_u

$$\{u\} = \{\bar{u}\}, \{W\}_x = \{\bar{W}\}_x. \tag{2.5}$$

Along s_σ

$$\{N\}_x = \{\bar{N}\}_x, \ \{V\}_x = \{\bar{V}\}_x \tag{2.6}$$

where

$$\{V\}_x{}^T = [-V_x \quad M_{xx}], \ \{N\}_x{}^T = [N_{xx} \quad N_{x\theta}]$$
$$\{W\}_x{}^T = [W \quad W_{,x}] \tag{2.7}$$

and

$$V_x = \frac{2}{R}M_{x\theta,\theta} + M_{xx,x} + N_{xx}W_{,x} + \frac{1}{R}N_{x\theta}W_{,\theta}. \tag{2.8}$$

In the light of above equations we now examine some variational formulations.

For thin circular cylindrical shells the minimum potential energy functional is given by

[1]

$$\Pi_p = \int_A \left\langle \frac{C}{2}\{\epsilon\}^T[C]\{\epsilon\} + \frac{D}{2}\{\kappa\}^T[C]\{\kappa\} - \bar{p}W \right\rangle \mathrm{d}A + \int_{s_\sigma} \langle \{\bar{V}\}_x{}^T\{W\}_x - \{\bar{N}\}_x{}^T\{u\} \rangle \,\mathrm{d}s. \tag{2.9}$$

In eqn (2.9) the first integral accounts for the strain energy of the cylindrical shell and the work of the prescribed distributed load $\bar{p}$ whereas the second integral accounts for the work of the forces prescribed along s_σ.

For the minimum potential energy principle, the admissible strains $\{\epsilon\}$ and curvatures $\{\kappa\}$ must satisfy certain compatibility conditions to ensure that the associated displacement fields are continuous and single valued. These admissibility conditions will be satisfied identically if strains and curvatures, entering eqns (2.9), are derived from the displacement fields via eqns (2.1). Furthermore, the admissibility conditions for the Π_p require that the displacements satisfy the kinematic boundary conditions in eqn (2.5).

3. STABILITY OF CIRCULAR CYLINDRICAL SHELLS

In order to obtain the stability condition of circular cylindrical shells we consider some equilibrium positions in the neighborhood of an initial equilibrium position. To this end we express the displacement, the moments and the membrane forces in two parts, as follows:

$$\{u\} = \{u\}_0 + \{u\}_1,\ W = W_0 + W_1 \tag{3.1}$$

$$\{N\} = \{N\}_0 + \{N\}_1,\ \{M\} = \{M\}_0 + \{M\}_1 \tag{3.2}$$

where $\{u\}_0$ and W_0 are the displacements corresponding to the initial position while $\{u\}_1$ and W_1 are the incremental displacements. $\{N\}_0$ and $\{M\}_0$ are also defined as the initial membrane forces and moments while $\{N\}_1$ and $\{M\}_1$ denote the incremental quantities.

Using the definitions (3.2) in the equilibrium equations (2.2) and eliminating the nonlinear quantities we obtain

Equilibrium equations

$$\begin{aligned} &[D]^T\{N\}_1 = 0 \\ &\{L\}^T\{M\}_1 + \{N\}_0\{W_1\}_{,xx} - \{R\}^T\{N\}_1 = 0 \end{aligned} \tag{3.3}$$

where we have presumed $\{W_0\}_{,x}$ are negligibly small and the initial force field $\{N\}_0$ and $\{M\}_0$ are in self-equilibrium, i.e.

$$\begin{aligned} &[D]^T\{N\}_0 = 0 \\ &\{L\}^T\{M\}_0 - \{R\}^T\{N\}_0 = -\bar{p}. \end{aligned} \tag{3.4}$$

If we substitute the definitions (3.1) into the kinematical relations (2.1) and assume that $\{W_0\}_{,x}$ is negligibly small, we obtain

$$\begin{aligned} &\{\epsilon\}_l = \{\epsilon\}_0 + \{\epsilon\}_1 \\ &\{\kappa\}_l = \{\kappa\}_0 + \{\kappa\}_1 \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} &\{\epsilon\}_0 = [D]\{u\}_0 + W_0\{R\},\ \{\epsilon\}_1 = [D]\{u\}_1 + W_1\{R\} \\ &\{\kappa\}_0 = \{W_0\}_{,xx}, \qquad \{\kappa\}_1 = \{W_1\}_{,xx}. \end{aligned} \tag{3.6}$$

Using the constitutive equations

$$\{N\}_0 = C[C]\{\epsilon\}_0,\ \{M\}_0 = -D[C]\{\kappa\}_0$$
$$\{N\}_1 = C[C]\{\epsilon\}_1,\ \{M\}_1 = -D[C]\{\kappa\}_1 \tag{3.7}$$

the potential energy functional may be expressed as follows:

$$\Pi_p = \Pi_p^0 + \Pi_p^1 + \Pi_p^2 \tag{3.8}$$

$$\Pi_p^0 = \int_A \left\langle \frac{C}{2}\{\epsilon\}_0^T[C]\{\epsilon\}_0 + \frac{D}{2}\{\kappa\}_0^T[C]\{\kappa\}_0 - \bar{p}W_0 \right\rangle \mathrm{d}A + \int_{s_\sigma} \langle \{\bar{V}\}_{0x}^T\{W_0\}_x - \{\bar{N}\}_{0x}^T\{u\}_0 \rangle\, \mathrm{d}s$$

$$\Pi_p^1 = \int_A \langle C\{\epsilon\}_0^T[C]\{\epsilon\}_1 + D\{\kappa\}_0^T[C]\{\kappa\}_1 \rangle \mathrm{d}A + \int_{s_\sigma} \langle \{V\}_{0x}^T\{W_1\}_x + \{\bar{V}\}_{1x}^T\{W_0\}_x - \{\bar{N}\}_{0x}^T\{u\}_{1x} - \{\bar{N}\}_{1x}^T\{u\}_0 \rangle \mathrm{d}s$$

$$\Pi_p^2 = \int_A \left\langle \frac{C}{2}\{\epsilon\}_1^T[C]\{\epsilon\}_1 + \frac{D}{2}\{\kappa\}_1^T[C]\{\kappa\}_1 + \frac{1}{2}\{\bar{N}\}_0[W_1]_x\{W_1\}_x \right\rangle \mathrm{d}A + \int_{s_\sigma} \langle \{\bar{V}\}_{1x}^T\{W_1\}_x - \{\bar{N}\}_{1x}^T\{u\}_1 \rangle \mathrm{d}s.$$

The variation of Π_p can now be written as follows;

$$\delta\Pi_p = \delta\Pi_p^1 + \delta\Pi_p^2 = 0$$

since $\delta\Pi_p^0 = 0$.

In $\delta\Pi_p^1$ the area integral is the variation in the strain energy of the shell at the initial equilibrium position resulting from the incremental displacements $\delta\{u\}_1$ and δW_1, and the boundary integral is the work done by the surface forces at the initial position of equilibrium (through the same incremental displacement). Hence since the position of the shell, which is characterized by the displacements $\{u\}_0$ and W_0, is a position of equilibrium we have

$$\delta\Pi_p^1 = 0 \tag{3.11}$$

which is merely the mathematical formulation of the principle of virtual displacements at the first of the two positions of equilibrium (see Novozhilov[7]).

Furthermore, using (3.11) in (3.10) we obtain

$$\delta\Pi_p^2 = 0. \tag{3.12}$$

This is the variational formulation of the problem of elastic stability.

The complementary energy functional Π_c^2 can now be easily derived from Π_p^2 by following the development outlined in Ref. [1]. If we do so, we obtain

$$\Pi_c^2(M,N,W) = -\int_A \left\langle \frac{1}{2C}\{N\}^T[C]^{-1}\{N\} + \frac{1}{2D}\{M\}^T[C]^{-1}\{M\} + \frac{1}{2}\{W\}_{,x}^T[N]_0\{W\}_{,x} \right\rangle \mathrm{d}A - \int_{s_u} \langle \{V\}_x^T\{\bar{W}\}_x - \{N\}_x^T\{\bar{u}\}_x \rangle \mathrm{d}s - \int_{s_\sigma} \langle (\{V\}_x^T - \{\bar{V}\}_x^T)\{W\}_{bx} - (\{N\}_x^T - \{\bar{N}\}_x^T)\{u\}_b \rangle \mathrm{d}s \tag{3.13}$$

where V_x in $\{V\}_x$ is the effective shear given by

$$V_x = \frac{2}{R} M_{x\theta,\theta} + M_{xx,x} + N^0_{xx} W_{,x} + \frac{1}{R} N^0_{\theta\theta} W_{,\theta}. \tag{3.14}$$

For the sake of simplicity we have dropped the subscript 1 for the incremental quantities. The moments $\{M\}$, membrane forces $\{N\}$ and W are to be varied in such a manner as to satisfy the equilibrium equations (3.3). To this end, let

$$\{N\} = \{\phi\}_{,xx}, \{M\} = [D_1]\{U\} - W\{N\}_0 + \phi\{R\} \tag{3.15}$$

where ϕ is the Airy stress function and $\{U\}^T = [U, V]$ are the Southwell stress functions[8], and

$$\{\phi\}^T_{,xx} = \left[\frac{1}{R^2}\phi_{,\theta\theta} \quad \phi_{,xx} \quad -\frac{1}{R}\phi_{,x\theta}\right]$$

$$[D_1] = \begin{bmatrix} 0 & \frac{1}{R}\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial x} & 0 \\ -\frac{1}{2R}\frac{\partial}{\partial\theta} & -\frac{1}{2}\frac{\partial}{\partial x} \end{bmatrix}. \tag{3.16}$$

In terms of the stress functions, Π_C^2 may be expressed as

$$\begin{aligned} \Pi_C^2(U, V, \phi, W) = & -\int_A \frac{1}{2C}\{\phi\}_{,xx}[C]^{-1}\{\phi\}_{,xx} + \frac{1}{2D}([D_1]\{U\} \\ & - W\{N\}_0 + \phi\{R\})[C]^{-1}([D_1]\{U\} - W\{N\}_0 + \phi\{R\}) \\ & + \frac{1}{2}\{W\}^T_{,x}[N]_0\{W\}_{,x} > \mathrm{d}A - \int_{s_u} \langle \{V\}_x^T\{\bar{W}\}_x \\ & - \{\phi\}_x^T\{\bar{u}\}\rangle \mathrm{d}s - \int_{s_\sigma} \langle (\{V\}_x^T - \{\bar{V}\}_x^T)\{W\}_{bx} \\ & - (\{\phi\}_x^T - \{\bar{N}\}_x^T)\{u\}_b\rangle \mathrm{d}s \end{aligned} \tag{3.17}$$

where

$$\{V\}_x = \begin{bmatrix} \frac{1}{R}\left(\frac{1}{R}U_{,\theta} + N^0_{x\theta}W\right)_{,\theta} - \phi_{,x} \\ \frac{1}{R}V_{,\theta} - N^0_{xx}W + \frac{1}{R}\phi \end{bmatrix} \tag{3.18}$$

From the variation of Π_C^2 we obtain the following Euler–Lagrange equations:

In the domain

$$\begin{aligned} &\delta\phi:\ \frac{1}{R^2}\epsilon_{xx,\theta\theta} - \frac{2}{R}\epsilon_{x\theta,x\theta} + \epsilon_{\theta\theta,xx} - \frac{1}{R}\kappa_{xx} = 0 \\ &\delta U:\ -\frac{1}{R}\kappa_{x\theta,\theta} + \kappa_{\theta\theta,x} = 0 \\ &\delta V:\ \frac{1}{R}\kappa_{xx,\theta} - \kappa_{x\theta,x} = 0 \\ &\delta W: N^0_{xx}(\kappa_{xx} - W_{,xx}) + 2N^0_{x\theta}\left(\kappa_{x\theta} - \frac{1}{R}W_{,x\theta}\right) + N^0_{\theta\theta}\left(\kappa_{\theta\theta} - \frac{1}{R^2}W_{,\theta\theta}\right) = 0. \end{aligned} \tag{3.19}$$

Equation (3.16) will be recognised as the compatibility conditions for circular cylindrical shells. The corresponding expressions along the boundaries of the cylinder are not given here for the sake of brevity (see [1]).

Next consider a procedure which complements Von Karman's formulation. The corresponding functional denoted, by Π_{MII}(i.e. second mixed functional) can be derived from Π_p^2 by a procedure outlined in Ref. [1]. The resulting functional takes the form

$$\Pi^2_{MII}(\epsilon, M, W) = \int_A \Big\langle \frac{C}{2}\{\epsilon\}^T[C]\{\epsilon\} - \frac{1}{2}\{W\}^T_{,x}[N]_0\{W\}_{,x} - \frac{1}{2D}\{M\}^T [C]^{-1}\{M\} - W\{R\}^T\{N\}\Big\rangle dA - \int_{s_\sigma} \langle(\{V\}_x^T - \{\bar{V}\}_x^T)\{W\}_{bx} + \{\bar{N}\}_x^T\{u\}\rangle ds - \int_{s_u} \{V\}_x^T\{\bar{W}\}_x\, ds. \tag{3.20}$$

Now let us satisfy the transverse equilibrium equations identically. To this end we write the moments $\{M\}$ in terms of some stress functions $\{U\}$ and transverse displacements W as well as a new set of variables defined below:

$$\{M\} = [D_1]\{U\} - W\{N\}_0 + \{\alpha\} \tag{3.21}$$

where

$$\{N\}_0 = \begin{bmatrix} N^0_{xx} \\ N^0_{\theta\theta} \\ N^0_{x\theta} \end{bmatrix}, \quad \{\alpha\} = \begin{bmatrix} \frac{C\nu}{R}\int u_x\, dx \\ C\int u_\theta\, d\theta \\ \frac{1}{2}\frac{C}{R}\int\int W\, dx\, d\theta \end{bmatrix} \tag{3.22}$$

Using the kinematical relations $(3.6)_2$ and the definition (3.21) in (3.20), we now express the functional Π^2_{MII} as follows:

$$\Pi^2_{MII}(u, U, W) = \int_A \Big\langle \frac{C}{2}([D]\{u\} + W\{R\})^T[C]([D]\{u\} + W\{R\}) - \frac{1}{2D}([D_1]\{U\} - W\{N\}_0 + \{\alpha\})^T[C]^{-1}([D_1\{U\} - W\{N\}_0 + \{\alpha\}) - \frac{1}{2}\{W\}_{,x}[N]_0\{W\}_{,x} - W\{R\}^T\{\alpha\}_{,x}\Big\rangle dA - \int_{s_\sigma} (\{V\}_x^T - \{\bar{V}\}_x^T)\{W\}_{bx} + \{\bar{N}\}^T\{u\}_x > ds - \int_{s_u} \{V\}x^T\{\bar{W}\}_x\, dS \tag{3.23}$$

where

$$\{\alpha\}^T_{,x} = \left[0 \quad \nu u_{x,x} + \frac{1}{R}u_{\theta,\theta} + \frac{W}{R} \quad 0\right]$$
$$V_x = -\frac{1}{R^2}U_{,\theta\theta} - \frac{1}{R}(N^0_{x\theta}W)_{,\theta} + \frac{C\nu}{R}u_x + \frac{C}{R^2}\int W\, dx \tag{3.24}$$

in $\{V\}_x$.

From the variation of Π^2_{MII} we obtain

In the domain A

$$\delta u_x\colon\; N_{xx,x}+\frac{1}{R}N_{x\theta,\theta}+\frac{Cv}{R}\left(\int\kappa_{xx}\,dx-W_{,x}\right)=0$$

$$\delta u_\theta\colon\; N_{x\theta,x}+\frac{1}{R}N_{\theta\theta,\theta}+C\left(\int\kappa_{\theta\theta}\,d\theta-\frac{1}{R^2}W_{,\theta}\right)=0$$

$$\delta W\colon\; N^0_{xx}(\kappa_{xx}-W_{,xx})+2N^0_{x\theta}\left(\kappa_{x\theta}-\frac{1}{R}W_{,x\theta}\right)$$

$$+N^0_{\theta\theta}\left(\kappa_{\theta\theta}-\frac{1}{R^2}W_{,\theta\theta}\right)+\frac{C}{R}\left(\iint\kappa_{x\theta}\,dx\,d\theta-\frac{1}{R}W\right)=0$$

$$\delta U\colon\; -\kappa_{\theta\theta,x}+\frac{1}{R}\kappa_{x\theta,\theta}=0$$

$$\delta V\colon\; -\frac{1}{R}\kappa_{xx,\theta}+\kappa_{x\theta,x}=0. \tag{3.25}$$

In eqns (3.25) we have the membrane equilibrium equations and the transverse compatibility equations. These are complementary to the Euler–Lagrange equations one obtains from Von Karman's formulation. It is worth noting that in (3.25) the coupling between the transverse and the membrane actions is imposed via some integrals which have a global, rather than a local character. In the next section we illustrate the use of Π_c and Π_{MII} for computation of buckling loads.

4. SOME ILLUSTRATIVE EXAMPLES

As a first illustrative example for the present two formulations, complementary and mixed II, consider the classical buckling analysis of a simply supported cylinder under uniform axial load.

(i) *The complementary formulation*

For this case, it is customary to assume

$$N^0_{xx}=-N,\; N^0_{\theta\theta}=0,\; N^0_{x\theta}=0. \tag{4.1}$$

The moments $\{M\}$ in terms of the stress functions U, V and ϕ, and W take the following form

$$M_{xx}=\frac{1}{R}V_{,\theta}+NW+\frac{1}{R}\phi,\; N_{xx}=\frac{1}{R^2}\phi_{,\theta\theta},\; M_{\theta\theta}=U_{,x}$$

$$N_{\theta\theta}=\phi_{,xx},\; M_{x\theta}=-\frac{1}{2}\left(\frac{1}{R}U_{,\theta}+V_{,x}\right),\; N_{x\theta}=\frac{1}{R}\phi_{,x\theta}. \tag{4.2}$$

We now choose

$$\phi=\beta_1\sin\lambda x,\quad U=\beta_2\cos\lambda x,\quad V=\beta_3,\quad W=\beta_4\sin\lambda x \tag{4.3}$$

where

$$\lambda=\frac{m\pi}{L},\; m=1,2,\ldots$$

By using these functions in the functional (3.17) we obtain

$$\Pi_C{}^2=[\beta_1\quad\beta_2\quad\beta_4]\begin{bmatrix}\frac{1}{R}+\frac{D}{C}\lambda^4 & \frac{v\lambda}{R} & \frac{N}{R}\\ & \lambda^2 & \lambda N\\ \text{sym} & & N^2\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & D(1-v^2)\lambda^2N\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_4\end{bmatrix}. \tag{4.4}$$

It remains to extremise the above form of $\Pi_C{}^2$ with respect to β_1, β_2 and β_4 to obtain the following non-conventional eigenvalue equation for N.

$$\begin{bmatrix} \dfrac{1}{R}+\dfrac{D}{C}\lambda^4 & \dfrac{\nu}{R}\lambda & \dfrac{N}{R} \\ & \lambda^2 & \nu\lambda N \\ \text{sym} & & N^2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ \text{sym} & & D(1-\nu^2)\lambda^2 N \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_4 \end{bmatrix} \tag{4.5}$$

or

$$N^2 \frac{(1-\nu^2)\dfrac{D\lambda^4}{Eh}}{\dfrac{1}{R^2}+\dfrac{D}{Eh}\lambda^4}\beta_4 = ND(1-\nu^2)\lambda^2\beta_4. \tag{4.6}$$

The eigenvalue N may be cancelled from both sides of the equation but, in so doing, we tacitly agree to discount zero and infinite values for N.

Finally, the critical value of N is obtained as

$$N = D\lambda^2 + \frac{Eh}{\lambda^2 R^2}. \tag{4.7}$$

(ii) *Mixed formulation II*

In this case we know that W is independent of θ. Thus, for the moments in (3.21) we write the following forms

$$\begin{aligned} M_{xx} &= \frac{1}{R}V_{,\theta} + \frac{C\nu}{R}\int u_x\,\mathrm{d}x + \frac{C}{R^2}\iint w\,\mathrm{d}x\,\mathrm{d}x + NW \\ M_{\theta\theta} &= U_{,x} + C\int u_\theta\,\mathrm{d}\theta \\ M_{x\theta} &= -\frac{1}{2}\left(\frac{1}{R}U_{,\theta} + V_{,x}\right). \end{aligned} \tag{4.8}$$

Considering the boundary conditions and using the constitutive equations, (4.8) becomes

$$\begin{aligned} M_{xx} &= \left[\frac{C\nu}{R\lambda}\beta_1 + \left(N - \frac{C}{R^2\lambda^2}\right)\beta_3\right]\sin\lambda x \\ M_{\theta\theta} &= -\lambda\beta_4\sin\lambda x \\ M_{x\theta} &= 0 \end{aligned} \tag{4.9}$$

where we have taken

$$\begin{aligned} &u_x = \beta_1\cos\lambda x, \quad u_\theta = \beta_2, \quad W = \beta_3\sin\lambda x, \\ &U = \beta_4\cos\lambda x, \quad V = \beta_5. \end{aligned} \tag{4.10}$$

Using eqns (4.9) and (4.10) in the functional Π^2_{MII} we obtain

$$\begin{aligned} \Pi^2_{MII} = R\iint\Bigg\{&\frac{C}{2}\left[u^2_{x,x} - \frac{1}{R^2}W^2\right] + \frac{1}{2}NW^2_{,x} - \frac{1}{2(1-\nu^2)D} \\ &\times [M^2_{xx} + M^2_{\theta\theta} - 2\nu M_{xx}M_{\theta\theta}]\Bigg\}\,\mathrm{d}x\,\mathrm{d}\theta = \frac{2\pi RL}{4}\Bigg\{\frac{C}{2}\left[\lambda^2\beta_1{}^2 - \frac{1}{R^2}\beta_3{}^2\right] \\ &+ \frac{1}{2}N\lambda^2\beta_3{}^2 - \frac{1}{2(1-\nu^2)D}\Bigg[\left(\frac{C\nu}{R\lambda}\beta_1 - N\beta_3 - \frac{C}{R^2\lambda^2}\beta_3\right)^2 \\ &+ \lambda^2\beta_4{}^2 + 2\nu\lambda\beta_4\left(\frac{C\nu}{R\lambda}\beta_1 + N\beta_3 - \frac{C}{R^2\lambda^2}\beta_3\right)\Bigg]\Bigg\}. \end{aligned} \tag{4.11}$$

Extemization of this functional in eqn (4.11) yields

$$\begin{bmatrix} (1-\nu^2)DC\lambda^2 - \dfrac{C^2\nu^2}{R^2\lambda^2} & -\dfrac{C\nu}{R\lambda}\left(N-\dfrac{C}{R^2\lambda^2}\right) & -\nu^2\dfrac{C}{R} \\ & \left(N-\dfrac{C}{R^2\lambda^2}\right)\left[(1-\nu^2)D\lambda^2-\left(N-\dfrac{C}{R^2\lambda^2}\right)\right] & -\nu\lambda\left(N-\dfrac{C}{R^2\lambda^2}\right) \\ \text{sym} & & -\lambda^2 \end{bmatrix} = 0 \quad (4.12)$$

from the determinant, (or on eliminating β_1 and β_4) we find

$$\frac{\left(N-\dfrac{C}{R^2\lambda^2}\right)^2(1-\nu^2)D\lambda^2}{\dfrac{-\nu^2C}{R^2\lambda^2}+D\lambda^2}\beta_3 = \left(N-\frac{C}{R^2\lambda^2}\right)(1-\nu^2)D\lambda^2\beta_3 \quad (4.13)$$

from which we obtain once again

$$N = D\lambda^2 + \frac{Eh}{R^2\lambda^2}. \quad (4.14)$$

It will be noted that results from the two formulations are coincident. This is because in both cases the exact solution is obtained. For loadings and boundary conditions for which exact admissible functions cannot be obtained, one will obtain different approximate solutions for the buckling loads and eigenmodes, for the two procedures.

For the second example, we consider a simply supported cylinder under uniform lateral pressure.

(i) *The complementary formulation*

In this case

$$N^0_{xx} = 0, N^0_{x\theta}, N^0_{\theta\theta} = -N \quad (4.15)$$

and we choose the stress functions U, V, and ϕ and the displacement W as follows:

$$\begin{aligned} U &= \beta_1 \cos\lambda x \sin n\theta, \quad W = \beta_3 \sin\lambda x \sin n\theta, \\ V &= \beta_2 \sin\lambda x \cos n\theta, \quad \phi = \beta_4 \sin\lambda x \sin n\theta. \end{aligned} \quad (4.16)$$

Using the expressions (4.16) in (3.15) and substituting them into Π_c^2 in (3.17), after some manipulations we obtain

$$\begin{aligned} \Pi_C^2 = -\frac{\pi RL}{2}\frac{1}{2(1-\nu^2)D}\Bigg\{ & \frac{1}{R^2}(\beta_4-n\beta_2)^2 + (N\beta_3-\lambda\beta_1)^2 \\ & -\frac{2\nu}{R}(\beta_4-n\beta_2)(N\beta_3-\lambda\beta_1) + \frac{1+\nu}{2}\left(\frac{n}{R}\beta_1+\lambda\beta_2\right)^2 \\ & + \frac{D}{C}\left(\frac{n^2}{R^2}+\lambda^2\right)^2\beta_4^2 - (1-\nu^2)D\frac{n^2}{R^2}N\beta_3^2 \Bigg\}. \end{aligned} \quad (4.17)$$

The extremisation yields

$$\begin{bmatrix} \lambda^2 + \dfrac{1+\nu}{2}\dfrac{n^2}{R^2} & \dfrac{1-\nu}{2}\dfrac{\lambda n}{R} & -\lambda N & \nu\dfrac{\lambda}{R} \\ & \dfrac{n^2}{R^2}+\dfrac{1+\nu}{2}\lambda^2 & \nu\dfrac{n}{R}N & -\dfrac{n}{R^2} \\ & & -(1-\nu^2)D\dfrac{n^2}{R^2}N + N^2 & -\dfrac{\nu}{R}N \\ \text{sym} & & & \dfrac{D}{C}\left(\dfrac{n^2}{R^2}+\lambda^2\right)^2 + \dfrac{1}{R^2} \end{bmatrix} = 0 \quad (4.18)$$

On eliminating β_1, β_2 and β_4 we find

$$\frac{N^2(1-\nu^2)D\dfrac{n^2}{R^2}}{\dfrac{R^2}{D}\dfrac{n^2}{(\bar{m}^2+n^2)^2}+\dfrac{n^2(n^2+\bar{m}^2)^2}{(1-\nu^2)C\bar{m}^4}}\beta_3=(1-\nu^2)D\frac{n^2}{R^2}N\beta_3, \tag{4.19}$$

or we then obtain

$$N=\frac{(\bar{m}^2+n^2)^2}{n^2}\frac{D}{R^2}\frac{\bar{m}^4}{n^2(n^2+\bar{m}^2)^2}(1-\nu^2)C. \tag{4.20}$$

(ii) *Mixed formulation II*

In this case we choose

$$\begin{aligned}
M_{xx}&=\frac{1}{R}V_{,\theta}+\frac{C\nu}{R}\int u_x\,\mathrm{d}x\\
M_{\theta\theta}&=U_{,x}+NW+C\iint W\,\mathrm{d}\theta\,\mathrm{d}\theta+C\int u_\theta\,\mathrm{d}\theta\\
M_{x\theta}&=-\frac{1}{2}\left(\frac{1}{R}U_{,\theta}+V_{,x}\right)
\end{aligned} \tag{4.21}$$

and for the stress functions and the displacement, we write

$$\begin{aligned}
&u_x=\beta_1\cos\lambda x\,\mathrm{sinn}\,\theta,\quad U=\beta_4\cos\lambda x\,\mathrm{sinn}\,\theta,\\
&u_\theta=\beta_2\sin\lambda x\cos n\theta,\quad V=\beta_5\sin\lambda x\cos n\theta,\\
&W=\beta_3\sin\lambda x\sin\theta
\end{aligned} \tag{4.22}$$

Π^2_{MII} becomes

$$\begin{aligned}
\Pi^2_{MII}=\frac{2\pi RL}{4}\Bigg\{&\frac{C}{2}\Bigg[\lambda^2\beta_1{}^2+\frac{n^2}{R^2}\beta_2{}^2-\frac{1}{R^2}\beta_3{}^2+2\nu\lambda\frac{n}{R}\beta_1\beta_2\\
&+\frac{1-\nu}{2}+\left(\frac{n^2}{R^2}\beta_1{}^2+2\frac{n}{R}\lambda\beta_1\beta_2+\lambda^2\beta_2{}^2\right)\Bigg]+\frac{1}{2}N\frac{n^2}{R^2}\beta_3{}^2\\
&-\frac{1}{2(1-\nu^2)D}+\Bigg[\left(-\frac{n}{R}\beta_5+\frac{C\nu}{R\lambda}\beta_1\right)^2+\left(-\lambda\beta_4+N\beta_3-\frac{C}{n^2}\beta_3+\frac{C}{n}\beta_2\right)^2\\
&-2\nu\left(-\frac{n}{R}\beta_5+\frac{C\nu}{R\lambda}\beta_1\right)\cdot\left(-\lambda\beta_4+\frac{C}{n}\beta_2+N\beta_3-\frac{C}{n^2}\beta_3\right)\\
&+2(1+\nu)+\frac{1}{4}\left(\frac{n}{R}\beta_4+\lambda\beta_5\right)^2\Bigg]\Bigg\}.
\end{aligned} \tag{4.23}$$

From the extremisation of this functional we obtain

$$\begin{bmatrix}
(1-\nu^2)DC\left(\lambda^2+\frac{1-\nu}{2}\frac{n^2}{R^2}\right)-\left(\frac{C\nu}{R\lambda}\right)^2 & (1-\nu^2)DC\frac{1+\nu}{2}\lambda\frac{n}{R}+\frac{C^2\nu^2}{R\lambda n} & \frac{C\nu^2}{R\lambda}\left(N-\frac{C}{n^2}\right) & -\frac{C\nu^2}{R} & \frac{C\nu}{R\lambda}\frac{n}{R}\\
 & (1-\nu^2)DC\left(\frac{n^2}{R^2}+\frac{1-\nu}{2}\lambda^2\right)-\frac{C^2}{n^2} & -\frac{C}{n}\left(N-\frac{C}{n^2}\right) & \lambda\frac{C}{n} & -\frac{C\nu}{R}\\
 & & \left(N-\frac{C}{n^2}\right)\left[(1-\nu^2)D\frac{n^2}{R^2}-\left(N-\frac{C}{n^2}\right)\right] & \lambda\left(N-\frac{C}{n^2}\right) & -\nu\left(N-\frac{C}{n^2}\right)\frac{n}{R}\\
 & & & -\left(\lambda^2+\frac{1+\nu}{2}\frac{n^2}{R^2}\right) & -\frac{1-\nu}{2}\frac{n}{R}\lambda\\
\text{sym} & & & & -\left(\frac{n^2}{R^2}+\frac{1+\nu}{2}\lambda^2\right)
\end{bmatrix} \tag{4.24}$$

or

$$\frac{\left(N - \frac{C}{n^2}\right)^2 (1 - \nu^2) D \frac{n^2}{R^2}}{\frac{R^2}{D} \frac{n^2}{(\bar{m}^2 + n^2)^2} + \frac{n^2(n^2 + \bar{m}^2)^2}{(1 - \beta^2) C \bar{m}^4}} \beta_3 = (1 - \nu^2) D \frac{n^2}{R^2} \left(N - \frac{C}{n^2}\right) \beta_3$$

then

$$N = \frac{(\bar{m}^2 + n^2)^2}{N^2} \frac{D}{R^2} + \frac{\bar{m}^4}{n^2(\bar{m}^2 + n^2)^2} (1 - \nu^2) C. \tag{4.25}$$

Once again we have coincident results from the two procedures since in both cases exact admissible functions were used in the extremisation.

CONCLUDING COMMENTS

In the foregoing we have introduced two new functionals for buckling analysis of thin circular cylinders. In the first all equilibrium requirements are satisfied *à priori* and the system equations emerge as a set of compatibility equations. In this sense we refer to this procedure as the complementary energy principle, in spite of the fact that the transverse displacement appears in this functional explicitly. In the second functional the transverse equilibrium and membrane compatibility equations are satisfied *à priori*. The Euler–Lagrange equations of this functional emerge as a set of membrane equilibrium equations and transverse compatibility equations. These equations are in a sense complementary to those that are obtained from Von Karman's formulation. However in the present case the system equations are integrodifferential in form.

The difference in the form of various functionals can be used to advantage in developing numerical methods of analysis. For instance, in a finite element procedure, the variables in Π_{MII} need only to satisfy C° continuity across element boundaries. It is well known that to satisfy the more stringent C^1 continuity explicitly one must employ high order polynomials. The major drawback of such finite element models is the appearance of not only the variables but also several orders of their derivatives as model variables. By contrast Π_{MII} allows one to use low polynomials as admissible functions for development of simple elements.

In finite element developments based on the potential energy functional, the admissible functions should include all six rigid body modes. This condition is rarely satisfied for cylindrical shell elements. The availability of alternate functionals implies that such requirements need not always be satisfied explicitly since they may be satisfied implicitly via the process of extremisation. Also it is important to note that in various formulations the errors of approximation will be confined to different sets of equations. Thus in the case of the potential energy functional all the errors of approximation will be confined to the equilibrium equations. By contrast in the complementary energy functional there will be no errors in the equilibrium equations. In this case all the errors are to be found in the compatibility equations.

Finally one can see that in a complete analysis two sets of variables are of interest. The first set which appear explicitly in the functional are usually computed relatively accurately. The second set, which are often obtained as derivatives of the first, are found to be less accurate as a consequence of differentiation. Thus in using the potential energy functional one can anticipate quite accurate results for displacements and less accurate results for the forces and moments. In the other functionals the primary and derived sets of variables are different and accordingly one can anticipate differing degrees of accuracies for variables of interest, from the other functionals.

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